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SUMMABILITY AND INTERPOLATION POLYNOMIALS

BY

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In a previous paper, [1], we used Bernstein polynomials to prove the following theorem:

Theorem 1. *Given a sequence* $\{s_n\}$, $0 \leqslant s_n \leqslant 1$, *for all* $n = 1, 2, \ldots$, *with an everywhere dense set of limit points in* $[0, 1]$ *and a function* $g(x)$ *of bounded variation over* $[0, 1]$ *with* $g(0) = 0$ *and* $g(1) = 1$, *there exists a regular matrix* $A = (a_{mn})$ *that sums* $\{s_n\}$ *and* $\{f(s_n)\}$ *for any function* $f(x)$ *continuous on* $[0, 1]$ *such that*

$$\lim_{m \to \infty} \sum_{n=1}^{\infty} a_{mn} f(s_n) = \int_0^1 f(x)\, d\, g(x).$$

As a consequence of the above theorem and a well-known theorem, it was pointed out that linear functionals in the space of continuous functions can be expressed in the form of a matrix.

Here we shall use a more general class of polynomials to prove that there exists a set, of unit measure, of regular matrices each of which possesses the property stated in theorem 1.

At first we introduce the definition and properties of the new set of polynomials. These polynomials appear in a paper due to G. GRÜNWALD, [2]. Their existence follows from the following theorem:

Theorem 2. *Let*

$$0 = \alpha_{n,0} < \alpha_{n,1} < \ldots < \alpha_{n,n} = 1, \quad n = 1, 2, \ldots,$$

be a sequence of points everywhere dense in $[0, 1]$. *Then there exists a matrix of polynomials*

$$A_{n,1}(x), A_{n,2}(x), \ldots, A_{n,n}(x), \quad n = 1, 2, \ldots,$$

each of which of degree $\leqslant n$ *and they posses the following properties*:

(i) $A_{n,k}(x) \geqslant 0$, $0 \leqslant x \leqslant 1$, $\sum_{k=1}^{n} A_{n,k}(x) \to 1$ *as* $n \to \infty$,

(ii) $A_{n,k}(\alpha_{n,k}) = 1$, $A_{n,k}(\alpha_{n,i}) = 0$ *for* $i \neq k$,

(iii) $I_n(x, f) = \sum_{k=1}^{n} A_{n,k}(x) f(\alpha_{n,k}) \to f(x)$ *uniformly for* $n \to \infty$, *for all functions* $f(x)$ *continuous on* $[0, 1]$.

These polynomials are called by GRÜNWALD: "the fundamental interpolation polynomials".

We shall use also the following theorem, [3] (P. 232):

Theorem 3. *Let $g(x)$ be a function of bounded variation on the closed interval $[a, b]$, and let $f_n(x)$ be a sequence of continuous functions on $[a, b]$, which converges uniformly to the (necessarily continuous) function $f(x)$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\, d\, g(x) = \int_a^b f(x)\, d\, g(x)$$

where the integrals on both sides are Riemann Stieltjes integrals.

Now we prove the following theorem:

Theorem 4. *Under the same assumptions of theorem* 1 *there exists a set, of unit measure, of regular matrices, each of which possesses the property stated in theorem* 1.

To prove this theorem we shall denote the sequence $\{\alpha_{n,k}\}$ by $\{\alpha_k\}$ where

$$\alpha_{\frac{1}{2}\mu(\mu+1)+p} = \alpha_{\mu+1,\,p}, \quad 1 \leqslant p \leqslant \mu+1; \quad \mu \geqslant 1 \ \&\ \alpha_1 = 1.$$

From the sequence $\{s_n\}$ we choose a subsequence $\{s_{n_k}\}$ such that

$$|s_{n_k} - \alpha_k| < \frac{1}{k}, \quad n_1 < n_2 < n_3 < \dots,$$

since both $\{s_n\}$ and $\{\alpha_k\}$ are dense everywhere.

We define an infinite matrix (a_{mn}) as follows:

$$a_{m,n_k} = \int_0^1 A_{m+1,p}(x)\, d\, g(x)$$

where

$$k = \frac{m(m+1)}{2} + p, \quad p = 1, 2, \dots, m+1$$

and

$$a_{m,n} = 0 \text{ for all other values of } n.$$

Now we prove that (a_{mn}) is a regular matrix.

$$\text{(1)} \qquad \sum_{n=0}^{\infty} a_{mn} = \sum_{p=1}^{m+1} \int_0^1 A_{m+1,p}(x)\, d\, g(x)$$

$$= \int_0^1 \sum_1^{m+1} A_{m+1,p}(x)\, d\, g(x)$$

$$\to \int_0^1 d\, g(x) = 1,$$

using property (i) and theorem 3.

$$\text{(2)} \qquad \sum |a_{m,n}| \leqslant \int_0^1 \sum_{p=1}^{m+1} A_{m+1,p}(x)|\, d\, g(x)| \leqslant \int_0^1 |d\, g(x)|, \quad \text{for all } m.$$

It remains to prove that:

$$\lim_{m\to\infty} a_{m,n}=0 \text{ for every } n=1, 2, \ldots. \tag{3}$$

we have:

$$a_{m,n_k}=\int_0^1 A_{m+1,p}(x)\, d\, g(x).$$

$$\therefore\ |a_{m,n_k}|<\int_0^1 A_{m+1,p}(x)\, |d\, g(x)|.$$

Let $\delta>0$ be choosen so small such that the variation of $g(x)$ over any interval of length 2δ is less than ε.

Now we have

$$\int_0^1 A_{m+1,p}(x)\, |d\, g(x)|$$

$$=\int_{|x-\alpha_{m+1,p}|<\delta} A_{m+1,p}(x)\, |d\, g(x)|+\int_{|x-\alpha_{m+1,p}|>\delta} A_{m+1,p}(x)\, |d\, g(x)|.$$

When δ is fixed and $m\to\infty$ then $A_{m+1,p}(x)=0$ if $|x-\alpha_{m+1,p}|>\delta$ and we have

$$|a_{m,n_k}|<\varepsilon_1(m)+\int_{|x-\alpha_{m+1,p}|<\delta} A_{m+1,p}(x)\, |d\, g(x)|$$

$$<\varepsilon_1(m)+\int_{|x-\alpha_{m+1,p}|<\delta} |d\, g(x)|<\varepsilon_1(m)+\varepsilon$$

$\therefore\ a_{m,n_k}\to 0$ as $m\to\infty$ for all n.

It follows that the matrix $(a_{m,n})$ satisfies the three Toeplitz conditions for regularity.

Secondly we show that

$$\lim_{m\to\infty} \sum a_{m,n_k} f(\alpha_k)=\int_0^1 f(x)\, d\, g(x).$$

We have:

$$\sum a_{m,n_k} f(\alpha_k)=\sum_{p=1}^{m+1}\int_0^1 f(\alpha_{m+1,p})\, A_{m+1,p}(x)\, d\, g(x)$$

$$=\int_0^1 \sum_{p=1}^{m+1} f(\alpha_{m+1,p})\, A_{m+1,p}(x)\, d\, g(x)$$

$$\to\int_0^1 f(x)\, d\, g(x),$$

using (iii) and theorem 3.

Finally we have

$$\lim_{k\to\infty} |f(s_{n_k})-f(\alpha_k)|=0.$$

So, remembering that

$$a_{m,n}=0,\ n\neq n_k,\ k=1, 2, \ldots,$$

we have

$$\lim_{m\to\infty}\sum_{n=1}^{\infty} a_{m,n}\, f(s_n)=\lim_{m\to\infty}\sum a_{m,n_k}\, f(s_{n_k})=\lim_{m\to\infty}\sum a_{m,n_k}\, f(\alpha_k)$$

and the theorem is proved.

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